ON THE DEVELOPMENT OF CONTINUOUS FUNCTIONS IN SERIES OF TCHEBYCHEFF POLYNOMIALS*

BY

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Introduction. Consider a system of orthogonal and normal Tchebycheff polynomials

$$\varphi_n(x) = a_n x^n + \cdots \qquad (n = 0, 1, 2, \cdots; \ a_n > 0)$$

corresponding to a certain interval (a, b) with the characteristic function $p(x)$ integrable and not negative on (a, b). Thus we have

$$\int_a^b p(x)\,\varphi_m(x)\,\varphi_n(x)\,dx = \begin{cases} 0, & m \neq n, \\ 1, & m = n. \end{cases}$$

For any function $f(x)$ we have a formal development as follows:

$$\text{(I)} \qquad f(x) \sim \sum_0^\infty A_i\,\varphi_i(x), \qquad A_i = \int_a^b p(x) f(x)\,\varphi_i(x)\,dx,$$

provided, of course, the right hand integrals exist.

Let us write

$$\text{(II)} \qquad f(x) = \sum_0^n A_i\,\varphi_i(x) + r_{n,f}(x) \equiv P_{n,f}(x) + r_{n,f}(x).$$

The question arises as to the convergence of the development (I) to $f(x)$ or—what is the same—the behavior of $r_{n,f}(x)$ in (II) for $n \to \infty$.

The case $(a, b) = (-1, 1)$, $p(x) = 1$ leads to Legendre's polynomials; it has been treated by Professor D. Jackson.†

In this paper we follow the method given by Professor Jackson in order to investigate the convergence of the development (I) involving *Tchebycheff's polynomials in general.* Hereafter, *the interval (a, b) is supposed to be finite, and $f(x)$ to be continuous on (a, b).*

1. $r_{n,f}(x)$ **expressed as a definite integral.** We obtain easily, using the formulas for A_i,

$$r_{n,f}(x) = f(x) - \int_a^b p(y) f(y) \sum_0^n \varphi_i(x)\,\varphi_i(y)\,dy,$$

* Presented to the Society, April 19, 1924.

† D. Jackson, *On the degree of convergence of a continuous function according to Legendre's polynomials*, these Transactions, vol. 13 (1912), pp. 305–318.

which gives, for $f(x) \equiv 1$,

$$1 = \int_a^b p(y) \sum_0^n \varphi_i(x)\varphi_i(y)\,dy.$$

Hence,

$$r_{n,f}(x) = \int_a^b p(y)\,[f(x) - f(y)]\,K_n(x, y)\,dy,$$

$$K_n(x, y) = \sum_0^n \varphi_i(x)\varphi_i(y);$$

$$r_{n,f}(x) = \frac{a_n}{a_{n+1}} \int_a^b p(y)\,[f(x) - f(y)]\,\frac{\varphi_{n+1}(x)\varphi_n(y) - \varphi_n(x)\varphi_{n+1}(y)}{x - y}\,dy,$$

since

$$K_n(x, y) = \frac{a_n}{a_{n+1}}\,\frac{\varphi_{n+1}(x)\varphi_n(y) - \varphi_n(x)\varphi_{n+1}(y)}{x - y}.*$$

Following Jackson's method we get for any polynomial $Q_n(x)$, of degree $\leqq n$,

$$0 = \int_a^b p(y)\,[Q_n(x) - Q_n(y)]\,K_n(x, y)\,dy,$$

$$r_{n,f}(x) = \int_a^b p(y)\,[\varphi(x) - \varphi(y)]\,K_n(x, y)\,dy,$$

$$\varphi(x) \equiv f(x) - Q_n(x).$$

We substitute here for $Q_n(x)$ a special polynomial, namely *the polynomial* $T_{n,f}(x)$ *of best approximation to* $f(x)$ *on* (a, b) (of degree n). Thus we get two formulas for $r_{n,f}(x)$:

(1) $$r_{n,f}(x) = \int_a^b p(y)\,[\varphi(x) - \varphi(y)]\,K_n(x, y)\,dy,$$

$$K_n(x, y) = \sum_0^n \varphi_i(x)\varphi_i(y); \quad \varphi(x) \equiv f(x) - T_{n,f}(x);$$

(2) $$r_{n,f}(x) = \frac{a_n}{a_{n+1}} \int_a^b p(y)\,[\varphi(x) - \varphi(y)]\,\frac{\varphi_{n+1}(x)\varphi_n(y) - \varphi_n(x)\varphi_{n+1}(y)}{x - y}\,dy.$$

Denote by $E_n(f)$ the *best approximation* on (a, b) of $f(x)$ by means of a polynomial of degree n, i. e.

(3) $$E_n(f) = \max |f(x) - T_{n,f}(x)| \text{ for } a \leqq x \leqq b.$$

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* Darboux, *Mémoire sur l'approximation des fonctions de très grands nombres,* Journal de Mathématiques Pures et Appliquées, ser. 3, vol. 4 (1878), pp. 5–60, 377–416; p. 413.

Using Schwarz's inequality, we get from (1)

$$|r_{n,f}(x)| \leqq 2\,E_n(f)\sqrt{\int_a^b p(y)\,dy}\;\sqrt{\int_a^b p(y)\,K_n^2(x,y)\,dy}$$
$$= 2\,E_n(f)\sqrt{\int_a^b p(y)\,dy}\;\sqrt{\sum_0^n \varphi_i^2(x)},$$

(4) $$|r_{n,f}(x)| \leqq 2\,Q\,E_n(f)\sqrt{K_n(x)} \qquad \left(Q^2 = \int_a^b p(y)\,dy\right),$$

(5) $$K_n(x) \equiv K_n(x,x) = \sum_0^n \varphi_i^2(x).$$

$E_n(f)$, as a function of n, has been investigated by Lebesgue, de la Vallée-Poussin, S. Bernstein, W. Stekloff, and in particular by D. Jackson.*

Table A: $E_n(f)$

Conditions imposed on $f(x)$	$E_n(f) =$	Author
1. $\lvert f(x_2)-f(x_1)\rvert \leqq \omega(\delta)$ for $\lvert x_2-x_1\rvert \leqq \delta$ $(a \leqq x_1,\, x_2 \leqq b)$.	$O\left(\omega\left(\frac{b-a}{n}\right)\right)$	D. Jackson
2. $\lvert f^{(p)}(x_2)-f^{(p)}(x_1)\rvert < \lambda\,\lvert x_2-x_1\rvert^\alpha$ (Lipschitz condition of order α $(a \leqq x_1,\, x_2 \leqq b,\ \lambda = \text{const.};$ $f^{(0)}(x) \equiv f(x))$).	$O\left(\frac{1}{n^{p+\alpha}}\right)$	"
3. $f^{(p)}(x)$ is continuous on (a,b).	$o\left(\frac{1}{n^p}\right)$	"
4. $\lvert f(x+\delta)-f(x)\rvert\cdot\lvert\log\delta\rvert<\lambda(=\text{const.})$	$O\left(\frac{1}{\log n}\right)$	"
5. $\lvert f(x+\delta)-f(x)\rvert\cdot\lvert\log\delta\rvert\to 0,\ \delta\to 0$ (Dini-Lipschitz condition).	$o\left(\frac{1}{\log n}\right)$	Lebesgue
6. $0<N<f^{(n)}(x)<M$ for $a \leqq x \leqq b$.	$\frac{2N}{n!} < \left(\frac{4}{b-a}\right)^n E_n(f) < \frac{2M}{n!}$	S. Bernstein; W. Stekloff
7. $f^{(p)}(x)$ exists for every p.	$n^p E_n(f) \to 0,\ n\to\infty$, for every p	S. Bernstein

We see from Table A that in order to evaluate $r_{n,f}(x)$ by means of (1, 2, 4), we need to know the order of $\varphi_n(x)$ or $K_n(x)$ with respect to n.

* D. Jackson, *Über die Genauigkeit der Annäherung stetiger Funktionen,* Dissertation, Göttingen, 1911, pp. 1–96.

2. **Order of K_n (x) (with respect to n).** It can be proved easily that

$$(6) \qquad \frac{1}{K_n(z)} = \min \int_a^b p(y)[1 + Z_1(y-z) + \cdots + Z_n(y-z)^n]^2 dy^*.$$

Therefore, using the notation $K_n(p;z)$, we conclude that

$$(7) \qquad \begin{array}{l} p_2(x) \geqq p(x) \geqq p_1(x) \text{ for } a \leqq x \leqq b \text{ implies} \\ K_n(p_2;z) \leqq K_n(p;z) \leqq K_n(p_1;z). \end{array}$$

On the other hand, to the characteristic function

$$(8) \qquad \begin{array}{ll} p_1(x) = (x-a)^{\alpha-1}(b-x)^{\beta-1}\Pi(x) & (\alpha, \beta > 0), \\ \Pi(x) \text{ a polynomial } (\Pi(a)\Pi(b) \neq 0), & \end{array}$$

there corresponds a special system of Tchebycheff's polynomials, a generalization of Jacobi's polynomials ($\Pi(x) \equiv 1$), and I have obtained the asymptotic expression for $K_n(p_1;z)$ at any point z in (a, b)†.

Thus we have

$$(9) \qquad \begin{array}{c} K_n(p_1;a) \sim n^{2\alpha}, \qquad K_n(p_1;b) \sim n^{2\beta}, \\ K_n(p_1;z) \sim n^{2m+1}, \end{array}$$

z being a root of $\Pi(x)$ of multiplicity $2m \geqq 0$ $(a+\varepsilon \leqq z \leqq b-\varepsilon)$‡. These results enable us to prove the following

THEOREM I. (i) *Suppose the point $x = z$ be inside the interval (a, b): $a+\varepsilon \leqq z \leqq b-\varepsilon$, and that there exist finite numbers $k > -1$, $A > 0$, c, d such that*

$$\frac{p(x)}{|x-z|^k} \geqq A \text{ for } (a \leqq) c \leqq x \leqq d \ (\leqq b) \qquad (c < z < d).$$

Let us take the smallest k possible satisfying the above conditions and $k = 0$ in case $p(z) > 0$. Then K_n $(p;z) = O$ $(n^{2k'+1})$, where k' is the smallest integer $\geqq k/2$. In particular K_n $(p;z) = O(n)$ for $k \leqq 0$.

(ii) *Suppose the point $x = z$ coincides with one of the end points of (a, b), say $z = a$. If*

$$\frac{p(x)}{|x-a|^k} \geqq A \qquad (k > -1, \ A > 0; \ a \leqq x \leqq c \ (\leqq b)),$$

* See my paper (where the proof is given for $z = 0$), Jacques Chokhate, *Sur le développement de l'intégrale $\int_a^b [p(y)/(x-y)]\,dy$ en fraction continue et sur les polynomes de Tchebycheff*, Rendiconti del Circolo Matematico di Palermo, vol. 47 (1923), pp. 25–46; p. 41.

† *On the asymptotitc properties of a certain class of Tchebycheff's polynomials*, read before the International Mathematical Congress, Toronto, August, 1924.

‡ Hereafter ε stands for an arbitrarily small but fixed quantity.

then

$$K_n\ (p;a) = O(n^{2k+2}).$$

Proof. (i) Consider the characteristic function corresponding to the interval (c,d) and defined as follows:

$$p_1(x) = A'(x-z)^{2k'} \text{ in } (c,\ d) \qquad (A'>0;\ c<z<d).$$

We have, then, A' being sufficiently small,

$$p(x) \geqq p_1(x) \text{ for } c \leqq x \leqq d.$$

Therefore (see 7,9), since

$$\min \int_a^b p(y)[1+Z_1(y-z)+\cdots+Z_n(y-z)^n]^2\,dy$$
$$\geqq \min \int_c^d p(y)[1+Z_1(y-z)+\cdots+Z_n(y-z)^n]^2\,dy,$$
$$K_n(p;z) \leqq K_n(p_1;z) = O(n^{2k'+1}), \qquad \text{Q. E. D.}$$

In a similar, slightly modified, way we prove the statement (ii) of our theorem. Formula (4) leads to the following

Corollary. *If $p(x)$ satisfies the conditions of Theorem* I, *then*

(i) $$|r_{n,f}(z)| < \tau E_n(f)\, n^{k'+1/2} \qquad (a+\varepsilon \leqq z \leqq b-\varepsilon),$$

(ii) $$|r_{n,f}(z)| < \tau E_n(f)\, n^{k+1} \qquad ((z-a)(z-b) = 0).*$$

3. **Order of $r_{n,f}(x)$ (with respect to n) (method of D. Jackson).** For any system of Tchebycheff's polynomials the following inequality holds:

(10) $$\frac{a_n}{a_{n+1}} < \frac{b-a}{2}.\dagger$$

Consider two cases:

Case I. *The point $x=z$ is inside the interval (a,b)*: $a<c<z<d<b$.

We write (1, 2) as follows:

(11) $$r_{n,f}(z) = \int_a^{c+\varepsilon} + \int_{c+\varepsilon}^{z-\varepsilon_n} + \int_{z-\varepsilon_n}^{z+\varepsilon_n} + \int_{z+\varepsilon_n}^{d-\varepsilon} + \int_{d-\varepsilon}^{b} = i_1+i_2+i_3+i_4+i_5,$$
$$\varepsilon_n > 0, \quad \varepsilon_n \to 0 \text{ for } n\to\infty.$$

* Hereafter we use τ to denote generally a fixed positive quantity, different, of course in different formulas, which does not depend on n.

† J. Chokhate, loc. cit., p. 33.

Here ε denotes a certain fixed positive quantity; n is supposed to be so large and ε so small that we have

$$\varepsilon_n < \varepsilon, \quad c + 2\varepsilon \leqq z \leqq d - 2\varepsilon.$$

Suppose the system of Tchebycheff's polynomials under consideration subjected to following conditions:

$$\left.\begin{array}{ll} (12) & p(x) < P \ (= \text{fixed const.}) \\ (13) & |\varphi_n(x)| < \tau n^\sigma \quad (n = 1, 2, \cdots; \ \sigma > -\tfrac{1}{2}^*) \end{array}\right\} (c \leqq x \leqq d),$$

where P, τ, σ do not depend upon x, nor upon n.

Consider first i_1 and i_5 in (11). Here we use formula (2), since $1/|x-y| < \varepsilon$. Using (3, 10) we get

$$|i_1|, |i_5| < \frac{b-a}{2} E_n(f) \left\{ |\varphi_{n+1}(z)| \int_a^b p(y) |\varphi_n(y)| \, dy + |\varphi_n(z)| \int_a^b p(y) |\varphi_{n+1}(y)| \, dy \right\};$$

$$(14) \qquad |i_1|, |i_5| < \tau E_n(f) n^\sigma,$$

assuming only the condition $|\varphi_n(z)| < \tau n^\sigma$ (since

$$\int_a^b p(y) |\varphi_i(y)| \, dy < \sqrt{\int_a^b p(y) \, dy}$$

by Schwarz's inequality). We use the same formula (2) to estimate i_2 and i_4.

Putting $z - y = u$, we get

$$|i_2|, |i_4| < \tau n^{2\sigma} E_n(f) \int_{\varepsilon_n}^{b-a} \frac{du}{u};$$

$$(15) \qquad |i_2|, |i_4| < \tau n^{2\sigma} E_n(f) |\log \varepsilon_n|,$$

under conditions (12, 13). In order to estimate i_3 in (11), we write

$$(16) \qquad i_3 = \int_{z-\varepsilon_n}^{z+\varepsilon_n} p(y) [\varphi(z) - \varphi(y)] \sum_0^n \varphi_i(z) \varphi_i(y) \, dy,$$

which gives

$$|i_3| < \tau E_n(f) \left[\varphi_0^2 + \sum_1^n i^{2\sigma} \right] \int_{z-\varepsilon_n}^{z+\varepsilon_n} p(y) \, dy;$$

$$(17) \qquad |i_3| < \tau E_n(f) n^{2\sigma+1} \varepsilon_n,$$

under conditions (12,13). If we replace condition (13) by a less restrictive one,

* $\sigma \leqq -\frac{1}{2}$ does not occur in applications (see below, p. 544).

(18) $$|\varphi_n(z)| < \tau n^\sigma \qquad (n = 1, 2, \cdots; \sigma > -\tfrac{1}{2}),$$

we can estimate i_2, i_4 and i_3 as follows:

Apply Schwarz's inequality to i_2, i_4 in (11). We get, since here $1/|z-y| \leq 1/\varepsilon_n$,

(19) $$|i_2|,\ |i_4| < \tau \frac{E_n(f)}{\varepsilon_n} n^\sigma$$

assuming only the condition (18). Assuming two conditions, (12) and (18), we get

$$|i_2| < \tau E_n(f) \sqrt{\int_{a+\varepsilon}^{z-\varepsilon_n} \frac{dy}{(z-y)^2}}\ \sqrt{\varphi_{n+1}^2(z) + \varphi_n^2(z)}\,;\text{*}$$

(20) $$|i_2|,\ |i_4| < \tau \frac{E_n(f)}{\sqrt{\varepsilon_n}} n^\sigma.$$

Similarly, applying Schwarz's inequality to i_3 in (16), we get

$$|i_3| < \tau E_n(f) \sqrt{\int_{z-\varepsilon_n}^{z+\varepsilon_n} p(y)\,dy}\ \sqrt{\int_a^b p(y) K_n^2(z, y)\,dy};$$

(21) $$|i_3| < \tau E_n(f) \sqrt{\varepsilon_n}\ \sqrt{K_n(z)} \quad \text{(under conditions (12))};$$

(22) $$|i_3| < \tau E_n(f) \sqrt{\varepsilon_n}\ n^{\sigma+1} \qquad \text{(under conditions (12, 18))}.$$

Case II. $(z-a)(z-b) = 0$; say $z = b$. Assume, as above,

(23) $$p(x) < P$$

(24) $$|\varphi_n(x)| < \tau n^\sigma \quad (n = 1, 2, \cdots; \sigma > -\tfrac{1}{2}) \Big\} ((a \leq) c \leq x \leq b),$$

or

(25) $$|\varphi_n(b)| < \tau n^\sigma \quad (n = 1, 2, \cdots; \sigma > -\tfrac{1}{2}).$$

Write (1,2) as follows:

$$r_{n,f}(b) = \int_a^{c+\varepsilon} + \int_{c+\varepsilon}^{b-\varepsilon_n} + \int_{b-\varepsilon_n}^{b} = i_1 + i_2 + i_3,$$

$$\varepsilon_n > 0, \quad \varepsilon_n \to 0 \text{ for } n \to \infty; \quad \varepsilon > 0; \quad \varepsilon_n < \varepsilon; \quad c + z\varepsilon < b.$$

Following the preceding discussion we estimate i_1, i_2, i_3 in a manner quite similar to that given above and find similar results.

We proceed to specify the infinitesimal ε_n. Take $\varepsilon_n = n^{-\beta}$, with $\beta > 0$, and choose β so as to make $r_{n,f}(z)$ of the highest order possible with respect to $1/n$. The results thus found (using the expressions above for i_1, i_2, i_3, i_4, i_5) are summarized in the following table.

* Similar inequality for $|i_4|$.

Table B: $r_{n,f}(x)$

Case	Conditions imposed on $p(x)$, $\varphi_n(x)$			$\|r_{n,f}(z)\|<\tau E_n(f)h_n$* with $h_n=$
	$p(x)$ is	$\varphi_n(x)<\tau n^\sigma$ for	with σ	
1. $a<c<z<d<b$	bounded for $c\leqq x\leqq d$	$c\leqq x\leqq d$	<0	n^σ (impossible; see below)
2. "	"	"	$0\leqq\sigma<\frac{1}{4}$	$n^{2\sigma}\log n$
3. "	"	"	$\sigma\geqq\frac{1}{4}$	$n^{\sigma+1/4}$
4. "	"	at the point $x=z$	$\sigma>-\frac{1}{2}$	$n^{\sigma+1/4}$
5. Same results hold in case $(z-a)(z-b)=0$ under analogous conditions imposed on $p(x)$ and $\varphi_n(x)$.				
6.	no conditions	at the point $x=z$	$\sigma>-\frac{1}{2}$	$n^{\sigma+1/2}$
7. any	no conditions			$\sqrt{K_n(z)}$†

The most interesting case is

$$(26)\qquad \sigma=0;\qquad |r_{r,f}(z)|<\begin{cases}\tau E_n(f)\log n & \text{(under conditions (12, 13)),}\\ \tau E_n(f)\,n^{1/4} & \text{(under conditions (12, 18)),}\\ \tau E_n(f)\,n^{1/2} & \text{(under condition (18)).}\end{cases}$$

The condition (18) with $\sigma=0$ holds, for instance, in the case of the characteristic function (8) (see second footnote on page 540) at any point $x=z$ $[(z-a)(z-b)\neq 0]$, where $\Pi(z)\neq 0$.

Another case, where we have (18) satisfied with $\sigma=0$, is given by G. Szegö.‡

It remains to prove that *it is impossible to have* (12, 13) *with* $\sigma<0$ (see Table B, case 1).

In fact, the contrary assumption gives

$$(27)\quad \frac{r_{n,f}(x)}{n^\sigma}\to 0\quad \text{for}\quad n\to\infty\quad\text{uniformly}\qquad (\sigma<0;\ c<c'\leqq x\leqq d'<d),$$

since $E_n(f)\to 0$ with $1/n$ for every continuous function.

Writing in general $E_n(f;a,b)$, we get, from the very definition of this quantity,

$$E_n(f;c',d')\leqq\max|r_{n,f}(x)|\quad\text{for}\quad c'\leqq x\leqq d'.$$

* τ is a fixed constant, not depending on n, nor on z (see (12, 13)).

† In some cases we know the order of $K_n(z)$, but not that of $\varphi_n(z)$.

‡ G. Szegö, *Über den asymptotischen Ausdruck von Polynomen*, Mathematische Annalen, vol. 86 (1922), pp. 114–140; p. 139.

Therefore, according to (27),

$$E_n(f;\, c', d') = o(n^\sigma) \quad \text{with} \quad \sigma < 0,$$

for every continuous function, which is impossible, because, as was established by D. Jackson,* for any $\sigma < 0$ there always exists a continuous function $f(x)$, for which

$$\frac{E_n(f;\, c', d')}{n^\sigma} \to 0 \quad \text{for} \quad n \to \infty, \quad \text{Q. E. D.†}$$

4. Order of $\varphi_n(x)$ (with respect to n).‡ We modify slightly our notations for Tchebycheff's polynomials and write in general, the characteristic function being $p(x)$,

$$\varphi_n(p;\, x) = a_n(p)\, x^n + \cdots \qquad (n = 0, 1, 2, \cdots;\ a_n(p) > 0). \tag{28}$$

We shall proceed to compare $\varphi_n(p;\, x)$ and $\varphi_n(q;\, x)$ corresponding to the same interval (a, b).

For this purpose consider

$$\begin{aligned}
\varDelta_{p,q} &= \int_a^b q(x)\, [\varphi_n(q;\, x) - \varphi_n(p;\, x)]^2\, dx = i_1 - 2 i_2 + i_3, \\
i_1 &= \int_a^b q(x)\, \varphi_n^2(q;\, x)\, dx = 1, \\
i_2 &= \int_a^b q(x)\, \varphi_n(p;\, x)\, \varphi_n(q;\, x)\, dx = \frac{a_n(p)}{a_n(q)}, \\
i_3 &= \int_a^b q(x)\, \varphi_n^2(p;\, x)\, dx = 1 + \int_a^b [q(x) - p(x)]\, \varphi_n^2(p;\, x)\, dx \\
&= 1 + \theta_1 \left(\frac{q-p}{p}\right)_{\max} \qquad (|\theta_1| \leqq 1),
\end{aligned} \tag{29}$$

where in general $(u)_{\max}$, $(u)_{\min}$ stand for the least upper and greatest lower bound respectively (or maximum and minimum) of $|u(x)|$ in (a, b).

We make use now of following inequalities:

$$\left(\frac{q}{p}\right)_{\min} < \frac{a_n^2(p)}{a_n^2(q)} < \left(\frac{q}{p}\right)_{\max}, \text{§}$$

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* Loc. cit., p. 56.

† But we may have $|\varphi_n(x)| < \tau n^\sigma$ with $\sigma < 0$ *at a certain point* $x = z$; e. g., for the polynomials of Jacobi (p. 540), with $\alpha, \beta < \frac{1}{2}$ at $x = a, b$ ($\sigma = \alpha - \frac{1}{2}$, $\beta - \frac{1}{2}$, respectively).

‡ The results of this paragraph are summarized in my article *Sur les polynomes de Tchebycheff*, Comptes Rendus, vol. 178 (1924), p. 2229. Here they are somewhat generalized.

§ J. Chokhate, *Sur quelques propriétés des polynomes de Tchebycheff*, Comptes Rendus, vol. 166 (1918), pp. 28-30.

which give

$$\left|\frac{a_n(p)}{a_n(q)}-1\right|<\left(\frac{q-p}{p}\right)_{\max}. \tag{30}$$

Thus we have, using (29),

$$i_2 = 1+\theta_2\left(\frac{q-p}{p}\right)_{\max} \qquad (|\theta_2|\leqq 1),$$

$$\Delta_{p,q} = \delta_n\left(\frac{q-p}{p}\right)_{\max} \qquad (0<\delta_n<3). \tag{31}$$

On the other hand, $Q_n(x)$ being an arbitrary polynomial of degree $\leqq n$, we have

$$Q_n(x) = \sum_0^n A_i\varphi_i(q;x), \qquad A_i = \int_a^b q(x)\,Q_n(x)\,\varphi_i(q;x)\,dx,$$

$$\begin{aligned} &|Q_n(x)| \leqq \sqrt{\sum_0^n A_i^2}\sqrt{\sum_0^n \varphi_i^2(q;x)};\\ &|Q_n(x)| \leqq \sqrt{\int_a^b q(x)\,Q_n^2(x)\,dx}\,\sqrt{K_n(q;x)},\ K_n(q;x)\equiv\sum_0^n\varphi_i^2(q;x).\end{aligned} \tag{32}$$

Apply (32) to the polynomial $\varphi_n(p;x)-\varphi_n(q;x)$ and use (31). We get

$$\left.\begin{aligned} &|\varphi_n(p,x)-\varphi_n(q;x)|<\tau\sqrt{\left(\frac{q-p}{p}\right)_{\max}}\sqrt{K_n(q;x)}\\ &|\varphi_n(p;x)-\varphi_n(q;x)|<\tau\sqrt{\left(\frac{q-p}{q}\right)_{\max}}\sqrt{K_n(p;x)}\end{aligned}\right\}\quad (0<\tau<\sqrt{3}). \tag{33}$$

Formulas (30, 33) lead to following

THEOREM II. *Suppose that $q(x)$, containing a parameter α, tends for $\alpha\to\alpha_0$ to $p(x)$ uniformly in (a, b), and that $p(x)\geqq p_{\min}>0$ in (a, b). Then $\varphi_n(q;x)\to\varphi_n(p;x)$ uniformly in (a,b), and $a_n(q)$ under the above conditions tends uniformly (with respect to n) to $a_n(p)$.*

Proof. ϵ being chosen as small as we please, take $|\alpha-\alpha_0|$ sufficiently small in order to give, for $a\leqq x\leqq b$,

$$|q(x)-p(x)|<\frac{p_{\min}}{2},$$

$$|q(x)-p(x)|<\frac{\epsilon^2 p_{\min}}{6K_n},$$

$$|q(x)-p(x)|<\epsilon p_{\min}, \qquad \text{for } a\leqq x\leqq b,$$

where $K_n = \max K_n(p; x)$ in (a, b). Then

$$q_{\min} > \frac{p_{\min}}{2}, \qquad \left(\frac{q-p}{p}\right)_{\max} < \frac{2(q-p)_{\max}}{p_{\min}},$$

$$|\varphi_i(p; x) - \varphi_i(q; x)| < \varepsilon \qquad (i = 0, 1, \cdots, n;\ a \leqq x \leqq b),$$

$$\left|\frac{a_n(p)}{a_n(q)} - 1\right| < \varepsilon \text{ for every } n, \qquad \text{Q. E. D.}$$

Consider first two special systems of Tchebycheff's polynomials:

$$(34)\quad \begin{aligned} &\varphi_n(p; x) \text{ with } p(x) = (x-a)^{\alpha-1}(b-x)^{\beta-1}\Pi(x), \\ &\qquad (\alpha, \beta > 0;\ \Pi(a)\Pi(b) \neq 0);\ \Pi(x) \text{ a polynomial of degree } s; \end{aligned}$$

$$(35)\quad \begin{aligned} &\varphi_n(q; x) \text{ with } q(x) = (x-a)^{\alpha-1}(b-x)^{\beta-1}, \\ &\qquad\qquad (\alpha, \beta > 0) \text{ (polynomials of Jacobi).} \end{aligned}$$

We have used these polynomials above (see pages 540, 544). We are now interested in finding what are the relations *between* $\varphi_n(p; x)$, $K_n(p; x)$ *and the degree* s *of* $\Pi(x)$ *in* (34).

For this purpose consider the development

$$(36)\quad \Pi(x)\varphi_n(p; x) = \sum_0^{n+s} A_i \varphi_i(q; x), \qquad A_i = \int_a^b p(x)\varphi_n(p; x)\varphi_i(q; x)\,dx,$$

where, as we see immediately,

$$(37)\qquad A_0 = A_1 = \cdots = A_{n-1} = 0.$$

On the other hand, as is well known,

$$(38)\quad \begin{aligned} &|\varphi_n(q; z)| < \tau \qquad (a + \varepsilon \leqq z \leqq b - \varepsilon), \\ &|\varphi_n(q; a)| < \tau n^{\alpha - 1/2}, \qquad |\varphi_n(q; b)| < \tau n^{\beta - 1/2}, \end{aligned}$$

where τ does not depend on z, nor on n.

Hence, (36, 37) give

$$\Pi(x)|\varphi_n(p; x)| \leqq \sqrt{\int_a^b p(x)\Pi(x)\varphi_n^2(p; x)\,dx}\ \sqrt{\sum_n^{n+s}\varphi_i^2(q; x)}$$

$$\leqq \sqrt{\Pi_{\max}}\ \sqrt{\sum_n^{n+s}\varphi_i^2(q; x)}.$$

Using (38), we get, n being sufficiently large (since $\varphi_n(q; x)$ does not depend on s):

$$\Pi(z)\,|\varphi_n(p;z)| < \tau\sqrt{s} \qquad (a+\varepsilon \leqq z \leqq b-\varepsilon), \tag{39}$$

$$\begin{gathered} |\varphi_n(p;z)| < \tau\sqrt{s} \qquad ((a<)\ c+\varepsilon \leqq z \leqq d-\varepsilon\,(<b);\ \Pi(x) \neq 0 \text{ in } (c,d)), \\ |\varphi_n(p;a)| < \tau\sqrt{s}\,n^{\alpha-1/2}; \qquad |\varphi_n(p;b)| < \tau\sqrt{s}\,n^{\beta-1/2}, \end{gathered} \tag{40}$$

where τ does not depend on z, nor on n, nor on s.

Formulas (40) answer the question stated above.

We return now to the general case. Assume that there exists a certain interval (c, d) such that

(41) *$p(x)$ is continuous and positive for $c \leqq x \leqq d$ $(a \leqq c;\ d \leqq b)$.*

Consider the polynomial $T_{m,p}(x)$ of best approximation to $p(x)$ in (c, d) of sufficiently large degree m. The polynomial $T_{m,p}(x)$ is also positive in (c, d). Now introduce the functions $\varphi_n(q; x)$, with

$$\begin{aligned} q(x) &\equiv T_{m,p}(x) \text{ in } (c,d), \\ &\equiv p(x) \text{ in } (a,c) \text{ and } (d,b). \end{aligned} \tag{42}$$

We can apply (33), which gives (since $(q-p)_{\max} = E_m(p)$)

$$|\varphi_n(p;x) - \varphi_n(q;x)| < \tau\sqrt{E_m(p)K_n(p;x)}, \tag{43}$$

where τ does not depend on x, nor on n, nor on m.*

We assume that m and n are increasing indefinitely, but $m/n \to 0$. Formula (43) combined with (40) (where $p(x)$, s, a, b must be replaced respectively by $q(x)$, m, c, d, and $\alpha = \beta = 1$, since $q(x) > 0$ for $c \leqq x \leqq d$) and with the results of Theorem I (page 540) gives, m and n being sufficiently large, the fundamental formula

$$\begin{gathered} |\varphi_n(p;z)| < \tau\left[\sqrt{m} + \sqrt{nE_m(p)}\right] \qquad (c+\varepsilon \leqq z \leqq d-\varepsilon), \\ |\varphi_n(p;c)|,\ |\varphi_n(p;d)| < \tau\sqrt{n}\left[\sqrt{m} + \sqrt{nE_m(p)}\right] \end{gathered} \tag{44}$$

under condition (41), *where τ does not depend on z, nor on n, nor on m, and $\varepsilon > 0$ is arbitrarily small, but fixed.*

In order to derive from (44) all the conclusions available, we take

$$m = \text{integral part of } n^{\beta} \text{ with } 0 < \beta < 1,\ n \to \infty.$$

* Formula (43) holds also for (a, b) infinitely large, provided (c, d) is finite.

Then

$$m \to \infty, \; n \to \infty, \; \frac{m}{n} \to 0, \; \frac{m}{n^\beta} \to 1, \; E_m(p) \to 0.$$

We can now use Table A (page 539), β being chosen so as to make the right-hand member in (44) of the highest order possible with respect to $1/n$. The results thus obtained are summarized in the following Table.

Table C: $\varphi_n(p; x)$

Conditions imposed on $p(x)$	z	$\varphi_n(p; z) =$
1. $p(x)$ is continuous and positive for $(a \leqq) c \leqq x \leqq d (\leqq b)$.	$c+\varepsilon \leqq z \leqq d-\varepsilon$	$o(n^{1/2})$
1, and 2. $\lvert p^{(k)}(x_2) - p^{(k)}(x_1) \rvert < \lambda \lvert x_2 - x_1 \rvert^\alpha$ for $c \leqq x_1, x_2 \leqq d$; $p^{(0)}(x) \equiv p(x)$	"	$O(n^{1/2(1+k+\alpha)})$*
(in particular for $k=0$, $\alpha=1$: Lipschitz condition).	"	$O(n^{1/4})$
1, and 3. $p^{(k)}(x)$ is continuous for $c \leqq x \leqq d$.	"	$O(n^{1/2(1+k)})$
1, and 4. $p(x)$ is indefinitely differentiable in (c, d).	"	$o(n^\sigma)$, $\sigma > 0$ arbitrarily small
5–8. Same conditions as in 1–4 above.	$a = c \leqq z \leqq d = b$	1/2 must be added to each of the exponents of n in 1–4 above.

The Tables A, B, C, *as well as the results of* § 2 (*concerning* $K_n(p; x)$) *enable us to determine the convergence and the order* (*with respect to* n) *of the remainder of the development of a continuous function into a series according to Tchebycheff's polynomials of a given type.*

Many theorems can be formulated in this way. As an illustration, we state the following:

THEOREM III. *Suppose* $f(x)$ *is continous in a given finite interval* (a, b) *and satisfies the condition*

$$|f(x_2) - f(x_1)| \leqq \omega(\delta) \text{ for } |x_1 - x_2| \leqq \delta \qquad (a \leqq x_1, \; x_2 \leqq b).$$

* See G. Szegö, loc. cit., p. 139.

Then, in the development

$$f(x) = \sum_0^n A_i \varphi_i(x) + r_{n,f}(x), \qquad A_i = \int_a^b p(x) f(x) \varphi_i(x)\, dx,$$

where $$p(x) = (x-a)^{\alpha-1}(b-x)^{\beta-1}\Pi(x) \qquad (\alpha, \beta > 0),$$

$$\Pi(x) \text{ a polynomial } (\Pi(a)\Pi(b) \neq 0),$$

we have

$$r_{n,f}(z) = O\left(\omega\left(\frac{b-a}{n}\right)\log n\right)^*$$

at any point $x = z$ *inside* (a, b), *provided* $\Pi(z) \neq 0$. *In particular, the development under consideration converges to* $f(x)$ *uniformly for* $c + \varepsilon \leqq x \leqq d - \varepsilon$ $(a \leqq c;\ d \leqq b;\ \varepsilon > 0$ *arbitrarily small, but fixed*), *if* $f(x)$ *satisfies a Dini-Lipschitz condition, provided the interval* (c, d) *contains no roots of* $\Pi(x)$.

In the particular case $\alpha = \beta = 1$, $\Pi(x) \equiv 1$, we get the results obtained by D. Jackson, as was mentioned above.

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* This follows from Table A_1, Table B_2 and p. 544.

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